

# FERMION DETERMINANTS 2003

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It is recommended that lattice QCD representations of the fermion determinant, including the discretization of the Dirac operator, be checked in the continuum limit against known QED determinant results. Recent work on the massive QED fermion determinant in two dimensions is reviewed. A feasible approach to the four-dimensional QED determinant with  $O(2) \times O(3)$  symmetric background fields is briefly discussed.

## 1 Introduction

The calculation of *every* physical process requires the gauge field measure

$$d\mu(A) = \frac{1}{Z} e^{-[(1/4) \int d^4x \text{Tr} F_{\mu\nu} F_{\mu\nu}]} \det(\not{D} + m) \prod dA_\mu^a(x), \quad (1)$$

obtained by making the Wick rotation to Euclidean space and integrating out the fermion fields. Here  $D_\mu = \frac{1}{i} \partial_\mu - g T^a A_\mu^a$ , where  $T$  is some real representation of a compact gauge group, and  $Z$  is the partition function. Granted its fundamental importance, it is curious that the fermion determinant,  $\det(\not{D} + m)$ , has generally failed to engage the attention of physicists outside lattice QCD since the 1970s and early 80s. Indeed, until recently<sup>1</sup> it was not even known what the strong coupling limit of  $\det$  with  $m \neq 0$  in QED<sub>2</sub> was for a general class of centrally symmetric background fields  $F_{\mu\nu}$ , though the question was asked twenty five years ago.<sup>2</sup> Part of the problem is that  $\det$  is a nonlocal function of  $F_{\mu\nu}$ . Moreover, it has to be known for general  $F_{\mu\nu}$  to say anything useful about the above measure. Consequently,  $\det$  is hard to calculate and so before the advent of large machines  $\det$  was expanded in a power series in  $g$ . Now there is nothing wrong with this provided the remainder after  $N$  terms,  $|\det - \sum_{n=0}^N \det_n g^n|$ , can be bounded. This requires nonperturbative information on  $\det$  that lattice versions can provide with  $\gtrsim 500$  GFLOP machines. Several lattice groups are now engaged in this endeavor.<sup>3</sup>

The following has to be kept firmly in mind: lattice representations of  $\det$ , which are taken to include the many lattice discretizations of the Dirac operator now in use, should be tested against known infinite-volume, continuum results for the fermion determinant to estimate any associated systematic computational error. Aside from such general results as the positivity of  $\det$  and the diamagnetic bound  $\det \leq 1$  in two and three dimensions<sup>4,5</sup> there is little to test  $\det$  against in the physical case  $m \neq 0$ . Therefore, we advocate the following remedy: test computations of  $\det$  against known continuum results for Abelian background fields in two, three, and four dimensions. This

would quantify discretization errors precisely for a particular representation of  $\det$ . Work in this direction has already begun.<sup>6</sup>

A survey of results for fermion determinants was presented at this workshop in 2001<sup>7</sup> and will not be repeated here. Instead, we will comment on some recent results found in.<sup>1</sup> We will conclude with a brief discussion of how progress can be made on the strong coupling limit of the fermion determinant in QED<sub>4</sub> for four-variable background fields.

## 2 Massive QED<sub>2</sub>

The one-loop effective Euclidean action  $S_{\text{eff}} = -\ln \det$  is calculated from the ratio  $\det(\not{D} - e\not{A} + m)/\det(\not{D} + m)$  of Fredholm determinants of Dirac operators. This ratio is formal and mathematical sense has to be made of it, such as Schwinger's proper time definition.<sup>8</sup> Starting from this definition a new representation of  $\ln \det$  for massive QED<sub>2</sub> was obtained,<sup>9</sup> namely

$$\begin{aligned} \frac{\partial}{\partial e} \ln \det &= \frac{e}{\pi} \int d^2 r \, \varphi \, \partial^2 \varphi \\ &+ 2m^2 \int d^2 r \, \varphi(\mathbf{r}) \langle \mathbf{r} | (H_+ + m^2)^{-1} - (H_- + m^2)^{-1} | \mathbf{r} \rangle, \quad (2) \end{aligned}$$

where the supersymmetric operator pair  $H_{\pm} = (\mathbf{P} - e\mathbf{A})^2 \mp eB$  are obtained from the Pauli operator  $(\mathbf{P} - e\mathbf{A})^2 - \sigma_3 eB$ ,  $A_{\mu} = \epsilon_{\mu\nu} \partial_{\nu} \phi$  and  $B = -\partial^2 \phi$ . The first term on the right-hand side of (2) is the contribution to the massive determinant from the massless Schwinger model.<sup>10</sup> By specializing to centrally symmetric, square-integrable magnetic fields with range  $R$ , (2) can be put into the form of a partial wave expansion:<sup>1</sup>

$$\begin{aligned} \frac{\partial}{\partial e} \ln \det &= \frac{e}{\pi} \int d^2 r \, \phi \, \partial^2 \phi \\ &- 2m^2 \sum_{l=-\infty}^{\infty} \int_0^R dr \, r \left( G_l^+(r, r; m e^{\frac{i\pi}{2}}) - G_l^-(r, r; m e^{\frac{i\pi}{2}}) \right) \phi(r) \\ &+ \frac{im^2}{\pi} \sum_{l=-\infty}^{\infty} e^{-i\pi|l|} \left( e^{2i\delta_l^+ \left( e^{\frac{i\pi}{2}} m \right)} - e^{2i\delta_l^- \left( e^{\frac{i\pi}{2}} m \right)} \right) \\ &\times \int_R^{\infty} dr \, r \ln \left( \frac{r}{R} \right) K_{|\frac{e\phi}{2\pi} - l|}^2(mr). \quad (3) \end{aligned}$$

Here  $G_l^{\pm}$  are outgoing-wave Green's functions for  $H_{\pm, l}$ ,  $\delta_l^{\pm}$  are the positive/negative chirality partial wave phase shifts, and  $\Phi$  is the total flux of  $B$ . Both  $G_l^{\pm}$  and  $\delta_l^{\pm}$  are analytically continued into the upper half  $k$ -plane by setting  $k = m e^{i\pi/2}$ . The representation (3) is exact. In order to penetrate

deeply into the nonperturbative regime we considered the limit  $mR \ll 1$  followed by  $|e\Phi| \gg 1$ . This is possible because the zero modes of  $H_{\pm,l}$  and their threshold resonance states can be calculated explicitly. For  $B(r) > 0$  with the two alternative sets of boundary conditions  $B(R) = 0$  or  $\lim_{r \rightarrow R^-} B'(r) < 0$  with  $B(R) > 0$ , the result in both cases is

$$\lim_{|e\Phi| \gg 1} \lim_{mR \ll 1} \ln \det = -\frac{|e\Phi|}{4\pi} \ln \left( \frac{|e\Phi|}{(mR)^2} \right) + O(|e\Phi|, (mR)^2 |e\Phi| \ln(|e\Phi|)). \quad (4)$$

The analysis in<sup>1</sup> of the remainder terms in (4) is not yet sharp enough to exclude the logarithm factor.

Several comments are in order. Firstly, we see that the presence of mass profoundly modifies the determinant. If  $m = 0$  *ab initio*, then the second term in (2) is absent and Schwinger's result is regained. But for  $m \neq 0$  and  $|e\Phi| \gg 1$  there is a build up of zero modes, and these eventually cancel the Schwinger contribution to  $\ln \det$ . In fact, the factor  $|e\Phi|/4\pi$  in (4) is proportional to the number of zero modes.

Secondly, the minus sign in (4) is a reflection of the paramagnetism of charged fermions in a magnetic field and is consistent with the diamagnetic bound  $\det \leq 1$ .<sup>4,5,11</sup>

Thirdly, there is a logarithmic dependence on  $e\Phi$ . This is not understood. The  $\text{QED}_2$  determinant is an entire function of  $e$  of order 2 for the class of fields considered.<sup>5,12</sup> The zeros of  $\det_{\text{QED}_2}$  are spread over the complex  $e$ -plane in quartets or in imaginary pairs. For  $e^2 \rightarrow \infty$  the zeros of  $\det_{\text{QED}_2}$  conspire to produce the logarithmic dependence on  $e\Phi$  seen in (4) so that the maximal growth of  $\det_{\text{QED}_2}$  is somewhere off the real axis of the complex  $e$ -plane. What is the physics behind this?

Finally, although the calculation leading to (4) was for a special class of magnetic fields the result only depends on a global property of  $B$ , namely  $\Phi$ . So it is possible that it also holds for noncentral, square-integrable magnetic fields. We know that the zero modes dominate in the limit considered and that their number only depends on  $e\Phi$  for any reasonable magnetic field.<sup>13</sup>

### 3 Duality

The continuation of a fermion determinant back to Minkowski space involves continuing the background field as well. This is nontrivial. In Euclidean space the calculation of  $\det_{\text{QED}_2}$  for centrally symmetric magnetic fields with rapid falloff reduces to a scattering problem with outgoing wave boundary conditions. Going back to Minkowski space and transforming the magnetic field to a one-dimensional electric pulse turns the problem into one of pair production with entirely different boundary conditions. This has been carefully investigated for the exactly solvable single-variable magnetic field  $B(x) = B \text{sech}^2(x/\lambda)$ <sup>14</sup> and for more general one-variable fields using a WKB

approach.<sup>15</sup> Assuming that the Wick rotation goes through in the two-variable case for the fields considered in the derivation of (4) we obtained<sup>1</sup>

$$2\pi \frac{\partial}{\partial m^2} S_{\text{eff}}^{\text{QED}_4}(E) = iL_1 L_2 \ln \det_{\text{QED}_2}(B \rightarrow e^{-i\pi/2} E) + \frac{L_1 L_2 \|E\|^2 e^2}{12\pi m^2}. \quad (5)$$

Here  $S_{\text{eff}}^{\text{QED}_4}$  is the  $\text{QED}_4$  one-loop Minkowski metric effective action obtained by making the duality transformation that takes the static magnetic field  $B(x_1, x_2) \mathbf{k}$  to the functionally equivalent electric pulse  $\mathbf{E}(x_3, t) = B(x_3, t) \hat{\mathbf{k}}$ .  $L_1$  and  $L_2$  are the sides of the space box in the  $x_1$  and  $x_2$  directions. The rules for making the analytic continuation we have outlined are found in Sec. VI of.<sup>1</sup>

Inserting the result (4) into the right-hand side of (5) gives the result

$$\frac{\partial}{\partial m^2} \text{Im} S_{\text{eff}}^{\text{QED}_4} < 0, \quad (6)$$

indicating that the pair production probability in the presence of  $E$  decreases with increasing fermion mass. This physically reasonable result depends on the minus sign in (4), which we have noted is a reflection of the Euclidean diamagnetic bound  $\det \leq 1$  in two dimensions. This bound is difficult to establish. It is first obtained on a lattice. Then the continuum limit has to be taken, and finally it has to be shown that the determinant so defined agrees with what physicists regard a determinant to be, such as Schwinger's proper time definition. It is reassuring that the diamagnetic bound has an immediate physical interpretation in Minkowski space.

Realistic laboratory electric pulses require the calculation of  $\det$  for more general magnetic fields than those considered so far. Then the question arises as to whether the Wick rotation to the associated dual electric pulse goes through. The answer should give us a better physical understanding of  $\det$ .

#### 4 $\text{QED}_4$

There are some major results. For example, if  $A_\mu \in L^p(\mathbb{R}^4)$ ,  $p > 4$ , then  $\det_{\text{QED}_4}$  can be expressed as a renormalized determinant that is an entire function of  $e$  of order 4 for massive fermions, even though  $F_{\mu\nu}$  may not be square integrable. This is summarized in reference<sup>12</sup> which refers the reader to the original sources. By an easy generalization of the two-dimensional case discussed in<sup>12</sup> to four dimensions the same result holds for massless fermions provided  $A_\mu \in L^p(\mathbb{R}^4)$  for some  $p$  in the open interval  $(\frac{4}{3}, 4)$ . If  $A_\mu$  is "winding", i.e., falls off as  $1/r$ , and the fermions are massless then the renormalized determinant defined in reference<sup>12</sup> ceases to exist, necessitating some other definition of the determinant. For massive fermions and  $A_\mu$  winding the renormalized determinant exists but requires knowledge of the Dirac operator's zero modes. These results also hold for  $\text{QCD}_4$ . For the special

case of background fields with  $O(2) \times O(3)$  symmetry the author has recently been able to calculate all the zero modes explicitly in  $\text{QED}_4$ . The symmetry  $O(2) \times O(3)$  reduces the problem of calculating  $\det_{\text{QED}_4}$  to a centrally symmetric scattering problem. The author sees no barrier to results for such fields in the near future.

## References

1. M.P. Fry, Phys. Rev. D **67**, 065017 (2003).
2. R. Balian, C. Itzykson, G. Parisi and J.B. Zuber, Phys. Rev. D **17**, 1041 (1978).
3. CP-PACS Collaboration, A. Ali Khan *et al.*, Phys. Rev. D **65**, 054505 (2002); D **67**, 059901(E) (2003);  
 QCDSF and UKQCD Collaborations, H. Stüben *et al.*, Nucl. Phys. B (Proc. Suppl.) **94**, 273 (2001);  
 MILC Collaboration, C. Bernard *et al.*, Nucl. Phys. B (Proc. Suppl.) **73**, 198 (1999);  
 SESAM Collaboration, N. Eicker *et al.*, Phys. Rev. D **59**, 014509 (1999);  
 UKQCD Collaboration, C.R. Allton *et al.*, Phys. Rev. D **60**, 034507 (1999).
4. D. Brydges, J. Fröhlich, and E. Seiler, Ann. Phys. (N.Y.) **121**, 227 (1979).
5. E. Seiler, in *Gauge Theories: Fundamental Interactions and Rigorous Results*, Proceedings of the International Summer School of Theoretical Physics, Poiana Brasov, Romania, 1981, edited by P. Dita, V. Georgescu, and R. Purice, Progress in Physics Vol. 5 (Birkhäuser, Boston, 1982), p. 263.
6. T.W. Chiu, Nucl. Phys. **B588**, 400 (2000); Phys. Rev. D **58**, 074511 (1998).
7. M. P. Fry, Int. J. Mod. Phys. A **17**, 936 (2002). As this reference contains a large number of printing errors the reader should consult the original version hep-th/0111244.
8. J. Schwinger, Phys. Rev. **82**, 664 (1951).
9. M. P. Fry, Phys. Rev. D **47**, 2629 (1993).
10. J. Schwinger, Phys. Rev. **128**, 2425 (1962).
11. D. Weingarten, Ann. Phys. (N.Y.) **126**, 154 (1980).
12. E. Seiler, Phys. Rev. D **22**, 2412 (1980).
13. Y. Aharonov and A. Casher, Phys. Rev. A **19**, 2461 (1979).
14. G. Dunne and T. Hall, Phys. Rev. D **58**, 105022 (1998).
15. G. Dunne and T. Hall, Phys. Rev. D **60**, 065002 (1999).